Reanalysis of nuclear spin matrix elements for dark matter spin-dependent scattering

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We show how to include in the existing calculations the corrections to the isovector coupling arising in chiral effective field theory recently found in Ref. [7]. The dominant effect can be taken into account by conveniently redefining the static spin matrix elements $\langle \mathbf{S}_{p,n} \rangle$: the largest one is reduced, on average, a 10%. To show the impact of these corrections we recalculate the limits on the WIMP-proton spin dependent cross scetion set by COUPP. We also give practical formulas to obtain $\langle \mathbf{S}_{p,n} \rangle$ given the structure functions in the various formalisms/notations existing in literature. We argue that the standard treatment of the spin-dependent cross section in terms of three independent isospin functions, $S_{00}(q)$, $S_{11}(q)$, $S_{01}(q)$, is redundant in the sense that the interference function $S_{01}(q)$ is the double product $|S_{01}(q)| = 2\sqrt{S_{00}(q)}\sqrt{S_{11}(q)}$.

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I. INTRODUCTION

The formalism for the cross section of the elastic scattering of weakly interacting massive particles (WIMP) off nuclei, was formulated many years ago for the neutralino [1, 2], a well motivated supersymmetric dark matter candidate [3]. The dominant interactions of a non-relativistic WIMP with a nucleus are the coherent spin-independent and spin-spin interactions [4, 5]. Nowdays many other particle physics models that predict a dark matter candidate have been proposed. However, in the non–relativistic limit, only a limited number of operators remain, as has been shown with the help of model independent effective filed theory [6]. To calculate the contribution to the cross section of these operators, the corresponding nuclear responses must be evaluated.

The scope of this paper is to clarify some previously unrecognized properties of the response functions of the spin dependent formalism and show how to take into account the effects of low energy pion physics recently found in Ref. [7].

In Section II and Appendix A we show that the interference function S_{01} is not an independent function but that it is determined by the isoscalar and isovector functions S_{00} and S_{11}.

In Section III we show how to account for the strong interactions correction and also compare, with the aid of Appendix B, the results of different calculations for nuclei of experimental interest.

To illustrate the impact of these effects we recalculate the limits on the WIMP-proton cross section using the last data of the COUPP experiment [8].

II. ONLY TWO FUNCTIONS

The effective axial interaction of a spin 1/2 Majorana fermion with nucleons

$$\mathcal{L} = 2\sqrt{2} G_F a_N \overline{\chi}\gamma^\mu\gamma^5\overline{\chi}\overline{N}\gamma_\mu\gamma^5 N$$

reduces in the non-relativistic limit to the operator $(2\sqrt{2}G_F)a_N 4m_\chi m_N(4\mathbf{S}_\chi \cdot \mathbf{S}_N)$ and leads to a WIMP-nucleus cross section that, after Refs. [1, 2], is given in the form

$$\frac{d\sigma}{dq^2} = \frac{8G_F^2}{v^2}\frac{S(q)}{2J+1}, \tag{1}$$

$$S(q) = a_0^2 S_{00}(q) + a_0 a_1 S_{01}(q) + a_1^2 S_{11}(q). \tag{2}$$

Although nuclear physics calculations are carried out in the isospin representation defined by the isoscalar coupling $a_0 = a_p + a_n$, and the isovector coupling $a_1 = a_p - a_n$, for practical purposes is natural to think in terms of neutrons and protons. The quantities usually employed by experimentalists are thus the couplings $a_p = (a_0 + a_1)/2$, $a_n = (a_0 - a_1)/2$, and the spin matrix elements of the protons and neutrons group in the nuclear ground state with total angular momentum J: $\langle \mathbf{S}_p \rangle \equiv \langle J, M = J| \sum_{i=1}^{Z} S_i^z |J, M = J\rangle$, $\langle \mathbf{S}_n \rangle \equiv \langle J, M = J| \sum_{i=1}^{A-Z} S_i^z |J, M = J\rangle$. In the protons-neutrons representation, Eq. (2) at zero momentum transfer reduces to

$$S(0) = \frac{2J+1}{\pi}\frac{J+1}{J}|a_p\langle \mathbf{S}_p\rangle + a_n\langle \mathbf{S}_n\rangle|^2. \tag{3}$$

The differential cross section is usually then rewritten as

$$\frac{d\sigma}{dE_R} = \frac{m_A}{2\mu_A^2 v^2}\sigma_A(0)\Phi^{SD}(q), \tag{4}$$

where the total WIMP-nucleus cross sections at $q = 0$ is

$$\sigma_A(0) = 8G_F^2\frac{\mu_A^2}{2J+1}S(0), \tag{5}$$

and the form factor normalized to one at $q = 0$

$$\Phi^{SD}(q) = \frac{S(q)}{S(0)}. \tag{6}$$

The functional form of the structure functions S_{ij} obtained from shell-model calculations is *exactly* a polynomial times an exponential in the dimensionless variable

$y = (qb/2)^2$ (or $u = 2y$), being b the oscillator size parameter, if the single particle states used to build the nuclear wave function are wave functions of the three-dimensional harmonic oscillator potential. This is not the case if the eigenfunctions of the Wood-Saxon potential are used [9–11] and when the effects discussed in Section III are included; anyway, it is always possible to *fit* them with a polynomial or an exponential times a polynomial in y or u. Hence in general

$$S_{ij} = e^{-2y} \sum_{k=0}^{k_{\max}} c_{ij}^{(k)} y^k, \text{ or } S_{ij} = \sum_{k=0}^{k_{\max}} c_{ij}^{(k)} y^k. \qquad (7)$$

It thus seems that three nuclear response function in the isospin representation S_{11}, S_{00}, S_{01} are necessary to furnish the cross section at finite momentum transfer, while only the two numbers $\langle \mathbf{S}_{p,n} \rangle$ are necessary at $q = 0$. This asymmetry is unnatural, $q = 0$ is just a value of the variable q, and actually it does not hold.

The functions $S_{00}(q)$, $S_{11}(q)$ are monotonic decreasing, always positive at every q, while S_{01} is always positive or always negative at every q. The simple reason behind this behavior, as shown in Appendix B, is that we can define two functions $S_0(q)$ and $S_1(q)$ (that maintain their sign) such that the isoscalar and the isovector functions S_{00} and S_{11} are nothing but the square of S_0 and S_1,

$$S_{00}(q) \equiv S_0^2(q), \quad S_{11}(q) \equiv S_1^2(q), \qquad (8)$$

and the interference function is the double product,

$$S_{01} \equiv 2S_0(q)S_1(q), \quad |S_{01}(q)| = 2\sqrt{S_{00}(q)S_{11}(q)}, \quad (9)$$

with sign $\rho_{01} = \mathrm{sgn}(S_{01})$ given by the relative sign between $S_0(q)$ and $S_1(q)$.

The physics at $q = 0$ is determined by the coefficients of order zero of the polynomials in Eq. (7),

$$S_{00}(0) = c_{00}^{(0)}, \; S_{11}(0) = c_{11}^{(0)}, \; \rho_{01} = \mathrm{sgn}(c_{01}^{(0)}). \qquad (10)$$

Comparing Eq. (2) evaluated at $q = 0$ with Eq. (3) we have

$$\langle \boldsymbol{S}_{p,n} \rangle^2 = \frac{\pi}{2J+1} \frac{J}{J+1} (S_{00}(0) + S_{11}(0) \pm S_{01}(0)),$$

being the plus sign associated with the protons and the minus sign with the neutrons. The static spin matrix elements, in the light of Eqs. (8), (9), (10), are then given by

$$\begin{aligned} \langle \boldsymbol{S}_p \rangle &= \lambda \sqrt{\frac{\pi}{2J+1} \frac{J}{J+1}} (\sqrt{c_{00}^{(0)}} + \rho_{01} \sqrt{c_{11}^{(0)}}), \\ \langle \boldsymbol{S}_n \rangle &= \lambda \sqrt{\frac{\pi}{2J+1} \frac{J}{J+1}} (\sqrt{c_{11}^{(0)}} - \rho_{01} \sqrt{c_{11}^{(0)}}), \end{aligned} \quad (11)$$

where λ is + or - in both, depending on the given nucleus.

III. INCLUDING STRONG INTERACTION CORRECTIONS

In Ref. [7] it has been shown that strong interaction effects due to pion exchange that arise in chiral perturbation theory renormalize the isovector coupling a_1. The authors also present a new shell-model calculation for the Xenon isotopes ^{129}Xe, ^{131}Xe giving numerical fits of the functions S_{ij} that include these new corrections.

Decomposing the spin-spin operator as a sum of the longitudinal and transverse electric operators projections of the axial current, see Appendix B, it is shown in Ref. [7] that the isovector coupling a_1 is renormalized as $\delta_{\mathcal{T}} a_1$ in $\mathcal{T}^{\text{el } 5}$ where

$$\delta_{\mathcal{T}} = 1 - 2\frac{q^2}{\Lambda} + \delta a_1, \qquad (12)$$

and $\delta_{\mathcal{L}} a_1$ in $\mathcal{L}^5$ with

$$\delta_{\mathcal{L}} = 1 - \frac{2 g_{\pi pn} F_\pi q^2}{2 m_N g_A (m_\pi^2 + q^2)} - \frac{2 c_3 \rho q^2}{F_\pi^2 (4 m_\pi^2 + q^2)} + \delta a_1. \quad (13)$$

The first q^2-dependent term of $\delta_{\mathcal{T}}$ and $\delta_{\mathcal{L}}$ arise at the one-body level (1b) currents, while the second q^2 dependent term in $\delta_{\mathcal{L}}$ and the momentum independent term

$$\delta a_1 = -\frac{\rho}{F_\pi^2} I (\frac{1}{3} k + \frac{1}{6 m_N}) \qquad (14)$$

is due to two-body (2b) currents. Here $F_\pi = 92.4$ MeV is the pion decay constant, $m_\pi = 138.04$ MeV the pion mass, m_N the nucleon mass, $g_{\pi pn} = 13.05$ the pion-nucleon coupling, $\Lambda = 1040$ MeV a cut-off scale, ρ the nuclear density, $\rho \in [0.10, 0.12]$ fm^{-3}, $I \simeq 0.58 - 0.60$ is a dimensionless factor. The largest uncertainties are in the parameters $c_3 \in [-2.2, -4.78]$ and $k = (2c_3 - c_4) \in [7.2, 14.0]$, see Ref. [12]. Considering the possible range in ρ, I and k, and calling for simplicity $\delta a_1 \equiv \delta$ we have for the interval and the average value:

$$\delta \in [-0.135, -0.314], \text{ and } \delta_{\text{av}} = -0.224. \qquad (15)$$

In principle, the calculation should be redone for all the nuclei of experimental interest. However we now show how to include them in existing calculations. The argument is the following.

The effect of the 2-body currents at $q = 0$ can be transferred into the static spin matrix elements by replacing a_1 in $S(0)$ with $a_1(1+\delta)$. Using the defining relations of $a_{0,1}$ we find

$$S(0)_{2\text{b}} = \frac{2J+1}{\pi} \frac{J+1}{J} |a_p \langle \mathbf{S}_p \rangle_{2\text{b}} + a_n \langle \mathbf{S}_n \rangle_{2\text{b}}|^2,$$

with

$$\begin{aligned} \langle \mathbf{S}_p \rangle_{2\text{b}} &= \langle \mathbf{S}_p \rangle + \frac{\delta}{2} (\langle \mathbf{S}_p \rangle - \langle \mathbf{S}_n \rangle), \\ \langle \mathbf{S}_n \rangle_{2\text{b}} &= \langle \mathbf{S}_n \rangle - \frac{\delta}{2} (\langle \mathbf{S}_p \rangle - \langle \mathbf{S}_n \rangle). \end{aligned} \qquad (16)$$

Obviously the total contribution of the proton and neutron groups to the total angular momentum does not change, $\langle \mathbf{S}_p\rangle_{2b} + \langle \mathbf{S}_n\rangle_{2b} = \langle \mathbf{S}_p\rangle + \langle \mathbf{S}_n\rangle$. Note that also the sign of the spin matrix elements is important in the determination of the size of the correction.

In Table I we report the spin values for the most important isotopes employed as detecting medium in actual experiments and their values after the inclusion of the 2-body current effects using the average value $\delta_{\rm av}$. The starting values are taken from papers that claim to present the best shell-model calculation for the given isotope (we rounded to 3 decimal digits where necessary).

The case of ^{19}F needs a particular comment. The experiments employing ^{19}F, COUPP [8], SIMPLE [13] and PICASSO [14], employ the spin values of Ref. [15], often quoting the compilation of Ref. [16], $\langle \mathbf{S}_p\rangle = 0.441$, $\langle \mathbf{S}_n\rangle = -0.109$. The successive refined shell-model calculation of Ref. [17] using the more realistic Wildhental interaction found $\langle \mathbf{S}_p\rangle = 0.4751$ and $\langle \mathbf{S}_n\rangle = -0.0087$. The protons contribution is thus similar but the neutrons contribution is now a factor $\simeq 50$ smaller and not 4 times smaller than the protons one.

The new calculations of Ref. [6] allows us to clarify the question. These authors use the same Wildhental interaction, the harmonic oscillator wave functions and the full $s-d$ shell structure, thus the results should agree with the ones of Ref. [17]. They furnish the analytical expressions of the response functions for ^{19}F but do not calculate the static spin values. We use Eq. (B3) to obtain $S_{ii}(0)$ and Eqs. (10), (11) to get the spin values. We find $\langle \mathbf{S}_p\rangle = 0.475(5)$ and $\langle \mathbf{S}_n\rangle = -0.008(68)$, in perfect numerical agreement with the values of Ref. [17]. Such an agreement is also found in the case of ^{23}Na, see Fig. 4 in Appendix B.

In Ref. [6] new shell-model calculations for ^{73}Ge, ^{127}I, ^{129,131}Xe are also given. We have verified that in these cases the spin values differ largely from the other known calculations, see also Ref. [19]. Given that the same authors in Ref. [6] state explicitly that these calculations are "exploratory", we do not use in Table I.

We see from Table I that with $\delta_{\rm av}$ the dominant spin matrix element (that determine if the isotope is sensitive to the spin-dependent WIMP-proton or WIMP-neutron cross section) typically suffer a 10% reduction while the value of the smaller spin matrix element increases but always remain more than one order of magnitude smaller than the dominant one.

At finite momentum transfer there are two energy scales to be compared. Experiments that measure the recoil energy typically restrict the search window below 100 keV to reduce the background. Using the oscillator size parameter [6, 7, 11]

$$b = \sqrt{\frac{41.467}{45A^{-1/3} - 25A^{-2/3}}}\ \mathrm{fm},$$

100 keV corresponds to $y \sim 0.07$ in the case of ^{19}F, $y \sim 0.39$ for ^{73}Ge and $y \sim 0.83$ in the case of ^{131}Xe. The other scale is set by the momentum transfer of the order of the pion mass, 100 MeV, that corresponds to $y \sim 0.2$, 0.28 and 0.38, respectively. q^2 dependent corrections can thus be neglected for light nuclei but should be considered in medium-heavy and heavy nuclei.

Table I. Spin matrix elements for the odd-mass isotopes employed in dark matter search experiments. The values of the last two columns include the 2-body currents corrections as given in Eq. (16) with $\delta = \delta_{\rm av}$ of Eq. (15).

Isotope (J) [Ref.]	$\langle \mathbf{S}_p\rangle$	$\langle \mathbf{S}_n\rangle$	$\langle \mathbf{S}_p\rangle_{2b}$	$\langle \mathbf{S}_n\rangle_{2b}$
^{19}F (1/2) [17]	0.475	-0.009	0.421	0.045
^{23}Na (3/2) [17]	0.248	0.020	0.222	0.045
^{29}Si (1/2) [17]	-0.002	0.133	0.013	0.118
^{73}Ge (9/2) [10]	0.030	0.378	0.069	0.339
^{127}I (5/2) [11]	0.309	0.075	0.283	0.101
^{129}Xe (1/2) [7]	0.010	0.329	0.046	0.293
^{131}Xe (3/2) [7]	-0.009	-0.272	-0.038	-0.242
^{133}Cs (7/2) [18]	-0.318	0.021	-0.280	-0.017

We anyway note that

$$\frac{S_{11}^{1b}(q)}{S_{11}^{1b}(0)} \simeq \frac{S_{11}^{2b}(q)}{S_{11}^{2b}(0)}, \tag{17}$$

as shown in Fig. 1 in the case of ^{131}Xe, blue full and dashed lines. The correction $2/\Lambda^2$ is practically negligible while the 1b correction in $a_{\mathcal{T}}$ is already included in the standard calculations [1, 2] in the form $1/(m_\pi^2 + q^2)$ using the Goldberger-Treiman relation $g_{\pi NN} \simeq g_A m_N / F_\pi$.

The normalized isospin functions have the property, as discussed in details in Refs. [20, 21],

$$\Phi_{00}(q) = \frac{S_{00}(q)}{S_{00}(0)} \simeq \Phi_{11}(q) = \frac{S_{11}(q)}{S_{11}(0)}. \tag{18}$$

For light nuclei they are practical identical for all the values of interest for the experiments, in heavy nuclei the difference get larger increasing q^2. This fact is consequence of the isospin symmetry that treats proton and neutrons on an equal footing. At small y, once normalized the isospin form factors behave like $1 - ay + by^2 + \mathcal{O}(y^3)$. In light nuclei as ^{19}F, ^{23}Na, ^{29}Si, the polynomials are of low order, typically up to fourth degree, thus this relations is respected on a larger range of y, that anyway cover the range of momentum transfer caused by dark matter elastic scattering. For medium heavy and heavy nuclei the more complicated structure requires higher order polynomials, from sixth to nine-th degree, thus the differences are more accentuated.

To conclude, the corrections can be taken into account in the standard expression of the cross section, Eq. (4),

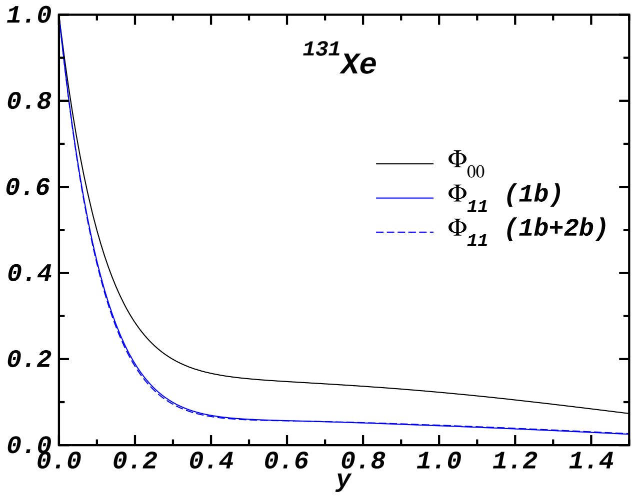

Figure 1. Isospin form factors for the isotope ^{131}Xe built with the functions of Ref. [7]. See Eq. (18) for definitions.

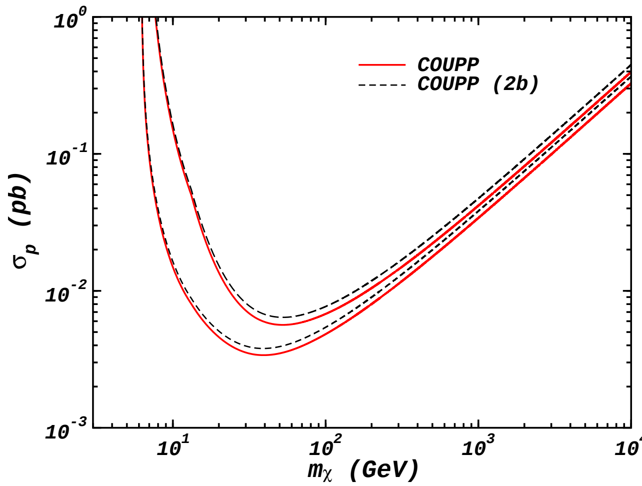

Figure 2. The solid red lines are the COUPP [8] 90% confidence level corresponding to two different models for the bubble nucleation efficiency of fluorine [8]. See the text for further details. The black dashed line are calculated with corrected spin values of Table I.

using the corrected static spin values $\langle \mathbf{S}_{p,n} \rangle_{2b}$ in $\sigma_A(0)$, Eq. (5), and using the known S_{ii} function in the form factor (6) normalizing them to one by factoring out their zero momentum transfer value.

In Fig. 2 we show the impact of the 2b currents on the limits of COUPP [8] (CF_3I). We use the standard Maxwellian distribution truncated at $v_{esc} = 544$ km/s, the velocity of the Sun $v_0 = 230$ km/s, the velocity of the Earth $v_E = 244$ km/s and $\rho_0 = 0.3$ GeV/cm^3 for the local dark matter density. The efficiencies, exposures and a threshold are specified in Ref. [8]. The two red curves correspond to two different models for the bubble nucleation efficiency of fluorine. Note that we find the agreement with the published curves using an upper limit on the number of events of $N^{UL} = 14.5$, that is the 90% confidence level limit obtained with the Feldman-Cousins method with 13 observed events (7 of 20 nuclear recoil events removed by the time isolation cut) and an expected background of 4.5.

The black dashed lines are the new limits calculated with $\delta_{\rm av}$. The reduction in the mass region above the minimum is about 10%. The effects should be included also in analysis concerning the impact of nuclear physics uncertainties on the reconstruction of the WIMP properties, mass and cross section, from the experiments with spin-dependent sensitivity [22].

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Appendix A: The isoscalar-isovector interference function

The spin-spin operator

$$O(q) = 4\mathbf{S}_\chi \cdot \sum_{i=1}^{A} \frac{1}{2}(a_0 \mathbb{1} + a_1 \tau_i^3)\mathbf{S}_i e^{-i\mathbf{q}\cdot\mathbf{r}_i} \tag{A1}$$

with $\mathbf{S}_i$ and $\mathbf{r}_i$ the spin and coordinates of the i–th nucleon $\tau_3|p\rangle = |p\rangle$, $\tau_3|n\rangle = -|n\rangle$ and $\mathbb{1}$, the identity operator in isospin space, is expressed in the standard formalism [1] (also in Ref. [6]) as the sum of the transverse electric and longitudinal projections of the axial current

$$O(q) = 4\hat{\mathbf{S}}_\chi \cdot (\mathcal{T}^{el\,5}(q)\hat{e}_\pm + \mathcal{L}^5(q)\hat{e}_0).$$

$\hat{e_0}$ is a spherical unit vector in the direction of the quantization axis taken to be one of $\mathbf{q}$ and $\hat{e}_\pm$ are the unit vector orthogonal to this direction. These operators are decomposed into multipoles as

$$\mathcal{T}^{el\,5} = \sum_{i=1}^{A} \frac{1}{2}(a_0 \mathbb{1} + a_1 \delta_\mathcal{T} \hat{\tau}_i^3)T(q), \tag{A2}$$

$$\mathcal{L}^5 = \sum_{i=1}^{A} \frac{1}{2}(a_0 \mathbb{1} + a_1 \delta_\mathcal{L} \hat{\tau}_i^3)L(q), \tag{A3}$$

where the detailed expressions for $T(q)$ and $L(q)$ are unnecessary for our argument. The coefficients $\delta_\mathcal{T}$ and $\delta_\mathcal{L}$ account for the effects discussed in Section III. Given the nuclear wave function $|A\rangle$, then

$$|\mathcal{M}|^2 \propto (|\langle A||\mathcal{T}^{el\,5}(q)||A\rangle|^2 + |\langle A||\mathcal{L}^5(q)||A\rangle|^2). \tag{A4}$$

There is no $\mathcal{T}^{el\,5} - \mathcal{L}^5$ interference in the modulus squared because of the orthogonality, but in Eq. (A4) each term

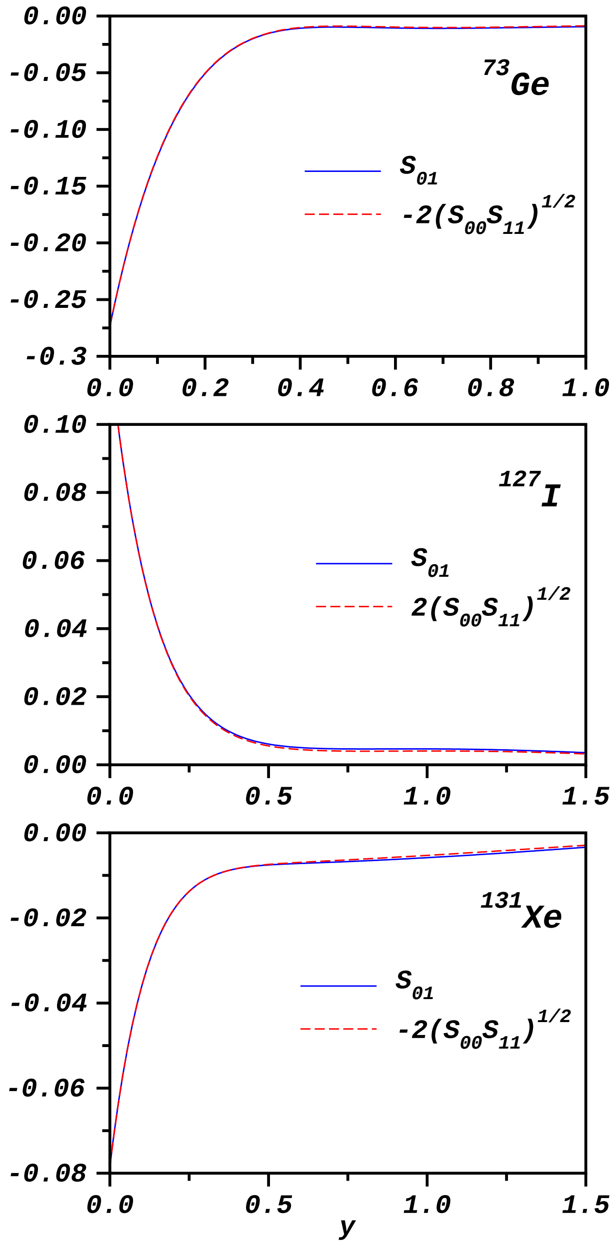

Figure 3. The isoscalar-isovector interference function S_{01} and the double product of Eq. (9). Functions for ^{73}Ge are from Ref. [10], for ^{127}I from Ref. [11] (Bonn A set), for ^{131}Xe from Ref. [7] (1b set).

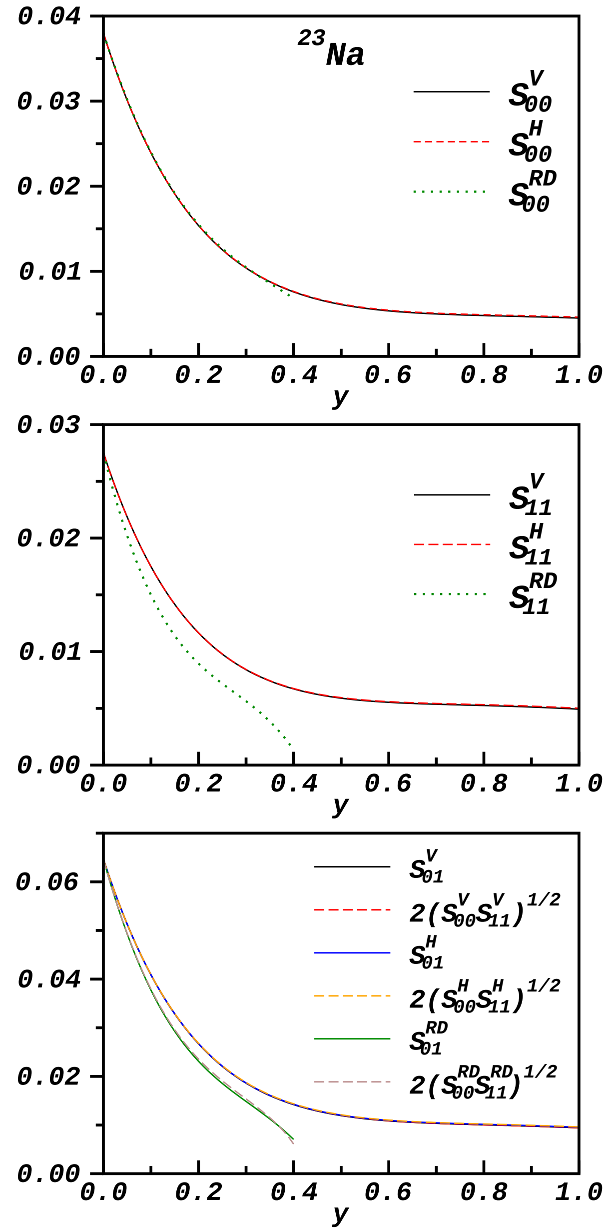

Figure 4. The structure functions for ^{23}Na. The superscript V and H indicate the that the functions are built from the functions of Ref. [17] and of Ref. [6] respectively, using the prescriptions of Appendix B. The superscript RD indicate the polynomial expressions of Ref. [11].

in Eq. (A4) gives contributions proportional to a_0^2, a_1^2 and the interference $a_0 a_1$. The functions $S_{ij}(q)$ arise as a sum of pieces after a rearrangement of these terms:

$$|\mathcal{M}|^2 \propto (a_0^2 S_{00}(q) + a_1^2 S_{11}(q) + a_0 a_1 S_{01}(q)). \tag{A5}$$

On the other hand, equivalently, the spin-spin operator can be decomposed, as done in Refs. [17],

$$O(q) = 4\mathbf{S}_\chi \cdot (a_0 \boldsymbol{S}_0 + a_1 \boldsymbol{S}_1),$$

where

$$\boldsymbol{S}_0(q) = \frac{1}{2}\sum_{i=1}^{A} \mathbb{1}\mathbf{S}_i e^{-i\mathbf{q}\cdot\mathbf{r}_i}, \tag{A6}$$

$$\boldsymbol{S}_1(q) = \frac{1}{2}\sum_{i=1}^{A} \hat{\tau}_i^3 \mathbf{S}_i e^{-i\mathbf{q}\cdot\mathbf{r}_i}.$$

are the nuclear spin operators associated with the isospin couplings. Performing the multipole decomposition on

these operators we are led to

$$|\mathcal{M}|^2 \propto |a_0\langle A||\boldsymbol{S}_0(q)||A\rangle + a_1\delta_1\langle A||\boldsymbol{S}_1(q)||A\rangle|^2. \quad \text{(A7)}$$

Now all the corrections are included in δ_1 and we can define two functions such that Eq. (A7) takes the form

$$|\mathcal{M}|^2 \propto (a_0 S_0^2(q) + a_1 S_1^2(q) + 2a_0 a_1 S_0(q) S_1(q)). \quad \text{(A8)}$$

Eq. (A5) and Eq. (A8) are equivalent: the isoscalar and the isovector functions S_{00} and S_{11} are squared of some functions $S_0(q)$ and $S_1(q)$ and the interference function is the double product $2S_0(q)S_1(q)$.

We explicitly show this fact in Fig. 3 for three isotopes used in actual experiments. The functions referring to ^{73}Ge are taken from Ref. [10], the ones for ^{127}I from Ref. [11] (Bonn A potential set) and for ^{131}Xe from Ref. [7]. See also Fig. 4, bottom panel, for ^{23}Na. The small differences that one can find in some cases can be ascribed to the intrinsic uncertainties of the numerical fits and numerical rounding.

Appendix B: Three equivalent notations

In the notation of Ref. [17] we can identify $S_0 = \Omega_0$, $S_1 = \Omega_1$. From these the normalized form factors are defined, $F_{ij}(q) = \Omega^2_{ij}(q)/\Omega^2_{ij}(0)$.

In the notation of Ref. [6] the basic spin response functions are, $N, N' = p, n$, $F_{\Sigma'}^{N,N'}$ (corresponding to the axial transverse electric operator $\mathcal{T}^{\mathrm{el}\ 5}$) and $F_{\Sigma''}^{N,N'}$ (corresponding to the axial longitudinal operator $\mathcal{L}^5$), thus

$$F_{44}^{N,N'} = \frac{1}{16}(F_{\Sigma'}^{N,N'} + F_{\Sigma''}^{N,N'}). \quad \text{(B1)}$$

The subscript "44" designate the spin-spin interaction. From these, the form factors in the isospin representation are

$$\begin{aligned} F_{44}^{00} &= \frac{1}{4}(F_{44}^{p,p} + F_{44}^{n,n} + 2F_{44}^{p,n}), \\ F_{44}^{11} &= \frac{1}{4}(F_{44}^{p,p} + F_{44}^{n,n} - 2F_{44}^{p,n}). \end{aligned} \quad \text{(B2)}$$

The standard isoscalar and isovector functions S_{00} and S_{11} read:

$$S_{ii}(q) \equiv \frac{2J+1}{16\pi}\Omega_i^2(0)F_{ii}(q) \equiv 4\frac{2J+1}{\pi}F_{44}^{ii}(q). \quad \text{(B3)}$$

This equivalence is illustrated in Fig. 3 for the isotopes ^{23}Na, that to our knowledge is the only nucleus for which the calculation in the three formalisms exists in literature. The results of Ref. [17] and Ref. [6] coincide exactly for both S_{00} and S_{11}. The isoscalar function of Ref. [11] agrees with the others two, while it is not the case for the isovector function. Note that these last ones are furnished as polynomial fits that are valid up to $y \sim 0.35$, for $y > 0.4$ they become negative. The double product rule is respected by the three calculations.

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